Concepts in Experimental Optics ACH 4074

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The Spiral Arc and the Rainbow

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1. Introduction

Ever since Isaac Newton observed the luminous pattern obtained when white light passed through a glass prism, physicists have been motivated to understand the effects taking place in experiments of the same type.

The knowledge gained from these studies has helped us to deepen the understanding of different aspects of the universe, as well as and motivated o development of some areas of mathematics. For example, research into field has led to advances in physics and mathematics. Knowledge acquired helped develop theories and models that are fundamental to many areas of science, including modern optics, electromagnetic theory and quantum physics. In addition, some of the most powerful tools in mathematical physics have been explicitly developed to deal with problems where some kind of related to reflection and as in the analysis of rainbow formation and problems refraction.

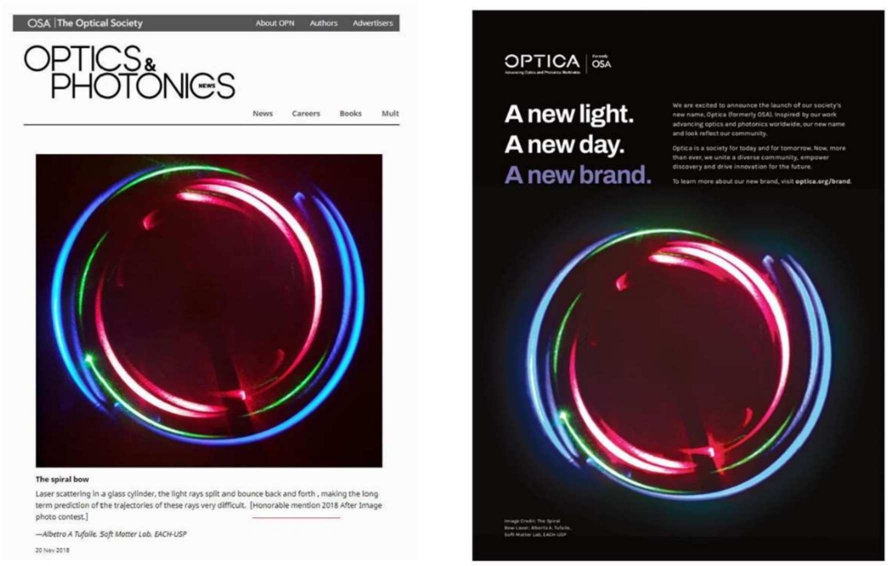

Figure 1 - The Spiral Bow light pattern.

Encouraged by this technique of light in a prism, we focus on a laser beam obliquely on a glass cylinder and obtained a beautiful pattern that we call the Spiralbow [1-5]. Understanding this type of optical phenomenon can lead to practical applications in technology and engineering, such as the manufacture of optical devices, detection systems, measuring instruments and advanced imaging technologies. By studying this experiment we can test the validity of different methods of explaining systems related to light interacting with lenses that are normally not very studied. In more detail, we have published some works on this system [6-8], and we strongly recommend the references cited in these two articles for a more complete overview of the subjects being discussed in this text. As this text was written for our Experimental Optics Concepts class, we have tried to avoid mathematical passages as much as possible, which can be developed appropriately at another time.

2. Geometric Optics

This type of cylinder/laser system is directly connected to the concepts of geometric optics due to the multiple reflections and refractions that occur in the glass cylinder when a laser beam is incident obliquely on it. When a laser beam is incident obliquely on the cylinder, part of the beam is reflected, part of the beam is refracted into the cylinder, undergoing multiple reflections inside the cylinder and escaping from the cylinder.

To explain what is happening in the geometrical optics of the cylinder/laser system, we'll start with a case that has been well explored in the literature, which is when light is incident normally on a cylinder, because the path of the rays is similar to what happens in a raindrop to form a rainbow. We will use *R* and *T* to differentiate the rays that occur in a sequence of interactions with the surface of the cylinder by reflection and transmission. Imagine a laser beam, called *I,* hitting the surface of the cylinder in a plane that is perpendicular to the axis of the cylinder. At this point, part of the light is reflected, represented by the ray *R,* while another part is refracted and enters the interior of the cylinder.

When this light hits the inner surface of the cylinder, the ray undergoes another refraction and leaves the cylinder, forming the *TT ray*. At this same point, a ray reflects internally and continues to propagate inside the cylinder until it reaches the interface again, leaving as the *TRT ray* by refraction.

As a parameter called h increases from zero to the maximum radius of the drop, the laser beam reaches an extreme value that creates a Cartesian ray. This type of ray is responsible for the formation of rainbows, in a similar way to what happens in an ordinary raindrop, but with the difference that in the case of the cylinder, it is the refractive index of the glass that plays the role of the refractive index of the water.

Let's analyze the scenario in which a laser beam is incident on the cylinder obliquely at an angle α_i to the normal, being contained in the meridional plane of the cylinder, which is defined by the plane passing through the axis of the cylinder. Following the law of reflection, we will have the reflected ray *R* at angle ar and a refracted ray that travels internally in the cylinder deflected by angle α_t with respect to the normal. When it reaches the other side of the inner surface of the glass cylinder, this ray splits into two, one internally reflected ray and the other *TT ray* refracted towards the outside of the cylinder at angle α_{tt} . The process is repeated with the *TRT* and *TRRT* rays. Thus, a ray that enters a cylinder with a specific inclination in relation to the axis will always come out with the same inclination, regardless of the sequence of refractions and reflections through which it passes.

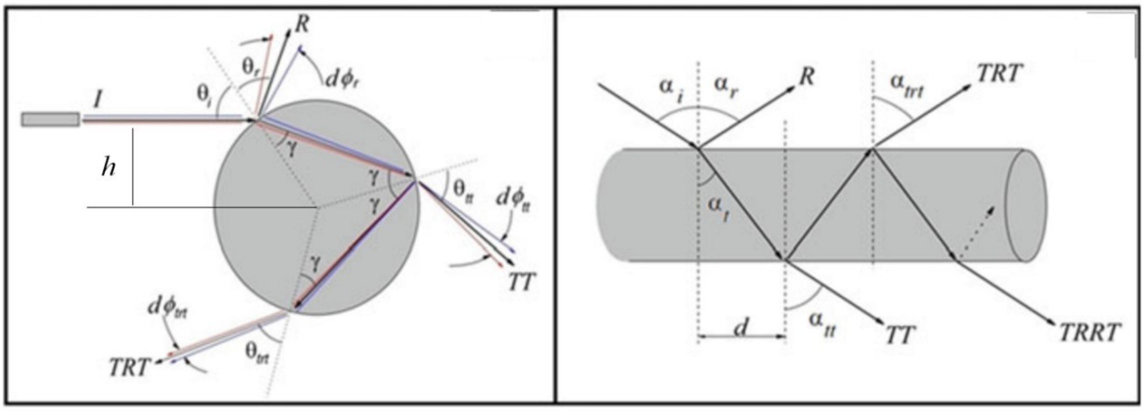

Figure 2 - Optical rays in a cylinder.

We will now consider the reflection and refraction of light in three dimensions. The laser beam is deformed when it hits the surface of the cylinder, even in the case of normal incidence of the light on the cylinder, because the curved surface of the cylinder causes a ray to be scattered over an angular range of dϕ., not just light points For cases where we have oblique incidence of the light on the glass cylinder, we have the formation of patterns around conical sections. For cases where the laser beam hits points on the surface of the cylinder that are outside the perpendicular plane or the meridional plane, the beams transmitted inside the cylinder undergo a sequence of reflections similar to those that occur in the dynamics of a particle inside a circle, associated with a helical-type translation as they advance towards the axis of the cylinder.

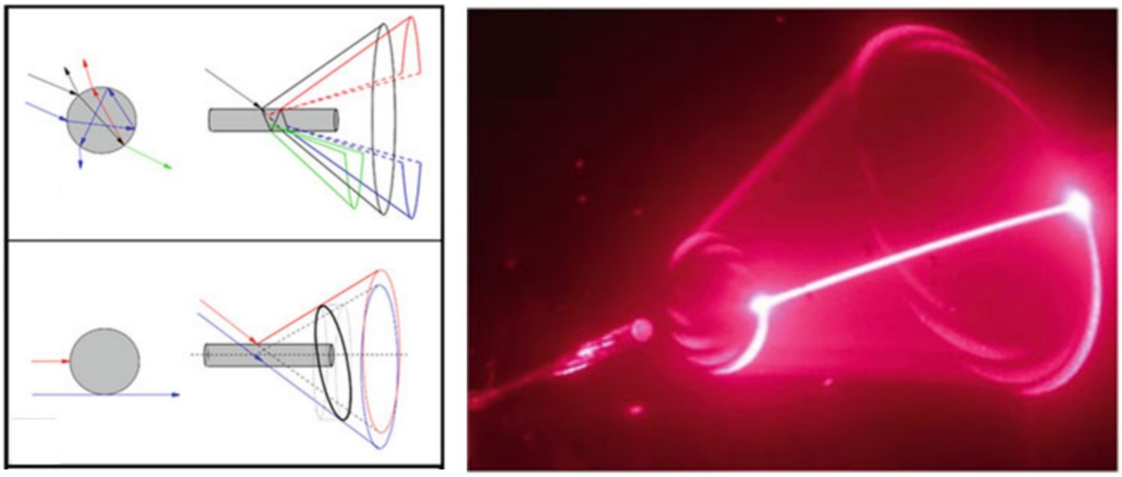

Figure 3 - The light cone when a laser beam hits a cylinder obliquely.

Considering a plane perpendicular to the axis of the cylinder where these rays are projected, the beams escaping from the cylinder begin to approach the axis at a spiral pattern, with decreasing intensities, due to the successive divisions of the beam each time it encounters the surface of the cylinder.

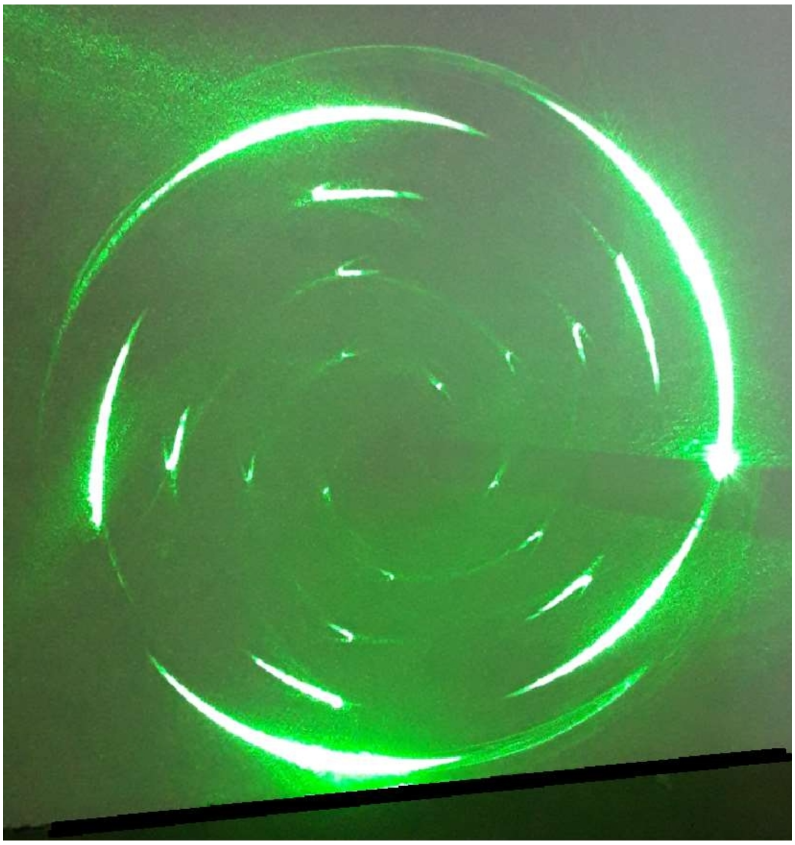

Figure 4 - A light spiral from the cylinder/laser experiment.

We can see an example of the experiment with the laser scattering obliquely on the glass cylinder compared to the ray diagram with the laser incident on the perpendicular plane of a cylinder or a sphere, with the laser projected onto a screen. This figure shows how rays with different interactions on the cylinder can diverge from a single laser point, forming luminous arcs related to parts of the conical projections of the light.

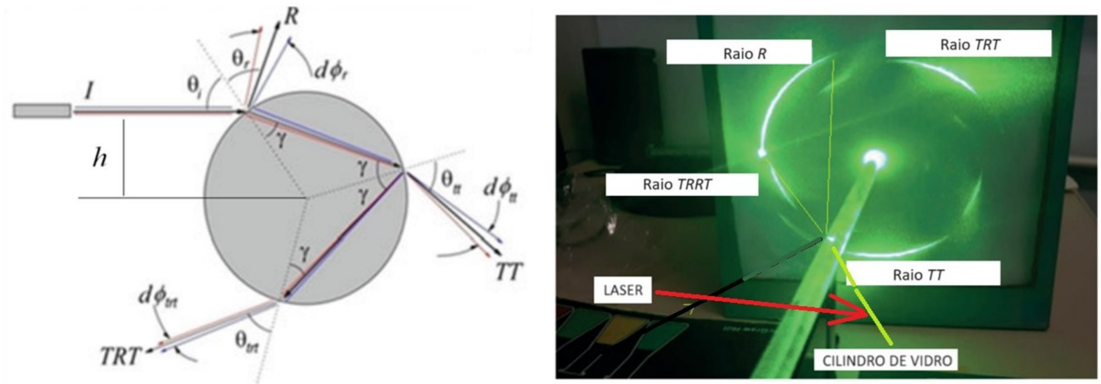

Figure 5 - Diagram of rays and radii in the laser/cylinder experiment.

3. Physical Optics

Another interesting point in this system is the concept of wave optics, because we observe the appearance of interference bangs in the Spiralbow pattern when parts of the laser beam overlap during its scattering by the glass cylinder. Therefore, when observing the pattern of light generated by the laser beam in the glass cylinder, it is possible to notice the appearance of interference bangs. This means that the light inside the cylinder behaves like a wave and produces areas where the light strengthens or cancels out, creating a characteristic Spiralbow pattern. In addition, depending on the relationship between the sizes of the diameter of the glass cylinder and the diameter of the laser beam, we can observe diffraction patterns, and this aspect of this physical system is the most studied from a historical point of view, i.e. diffraction by circular cylinders and spheres are problems of optics and acoustics, also called canonical problems of physics. This means that these problems have been widely studied and serve as fundamental cases for understanding and analyzing diffraction in physical systems, allowing for a valuable understanding of the behavior of light or sound when interacting with geometric objects such as cylinders and spheres. The exact nature of the diffraction patterns depends on the specific proportions between the sizes of the structures and the properties of the incident waves.

According to the literature, Lord Rayleigh was the first to solve the problem of wave scattering by an insulating circular cylinder, while Heinrich Lorenz investigated wave scattering by an insulating sphere. J.J. Thomson also contributed to the study of scattering in conductive materials with these geometries.

The case of wave scattering on spheres is known as "Mie scattering" in honor of Gustav Mie, who published a comprehensive article on the subject in 1908. Subsequently, over the years, new solutions with simpler and more concise mathematical formulations were developed for these problems, and these solutions are found in many modern electromagnetism textbooks. These historical contributions have played a fundamental role in understanding the behavior of electromagnetic waves when they interact with objects of different geometric shapes.

The scientific method applied in these studies tries to answer the question of what shape the electromagnetic wave is and how it propagates when an electromagnetic wave interacts with an obstacle. In this method, we start from Maxwell's equations of electromagnetism and find the solutions to the differential equations related to each geometry, applying the relevant initial or boundary conditions.

In studies of the interaction of electromagnetic waves with obstacles of different geometries, the types of solutions found depend on the specific characteristics of the problem. However, there are some general types of solutions that can be obtained with analytical, numerical, approximate and experimental solutions.

For example, in some cases it is possible to find analytical solutions, which are expressed in terms of well-defined mathematical functions. These solutions provide an exact description of the behavior of electromagnetic waves in the presence of the obstacle. However, obtaining analytical solutions is generally only feasible for simple, symmetrical geometries. For more complex geometries or problems that cannot be solved analytically, numerical methods are used. This involves discretizing space and time and solving the equations iteratively, often with the aid of computers. Numerical solutions offer an approximate description of wave behavior and allow more complex problems to be modeled. However, in many cases it is difficult or impossible to obtain exact or precise numerical solutions. Therefore, approximation techniques are used, such as series expansions or perturbative methods. These approximate solutions can be useful for obtaining qualitative information about the problem. Furthermore, in practical situations, such as laboratories or field experiments, it is often more convenient to obtain experimental solutions. This involves directly measuring the behavior of electromagnetic waves in a real environment. Experimental solutions are used to validate theoretical models and simulations. When all this fails to provide the relevant solutions, theory is combined with experimental data to obtain hybrid solutions. This approach can be valuable for considering the impact of real-world imperfections that may not be captured by purely theoretical models.

4. Rainbows and Catastrophe

As mentioned earlier, the pattern seen in the Spiralbow is closely related to the formation of rainbows. This is because rainbows originate from the creation of light rays that generate caustics at specific angles of observation of the light that is refracted and reflected. Caustics represent areas where light concentrates or focuses on particular points or lines. These points or lines play a fundamental role in the formation of the rainbow, since they determine the colors and the angles at which the colors are perceived.

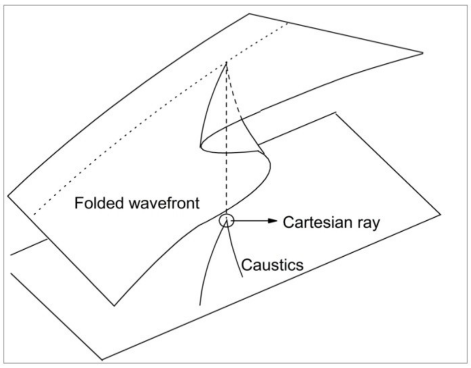

Figure 6 - The fold catastrophe, Cartesian radius and caustics.

To study the formation of these caustics, we consider the tracing of light rays in a sectional cut made by a plane that passes through the center of a spherical drop, which gives us a circumference, which is identical to the cross-section of the glass cylinder. On the other hand, when we analyze the propagation of the wave on this same circumference, we see the formation of bends in this light wave exactly in the region where we have the formation of Cartesian rays in geometric optics. This type of bend in a surface was studied in the "theory of catastrophes" by René Thom, who also classifies them mathematically as "caustics", which are related to mathematical curves known as involutes. When the laser beam hits the glass cylinder perpendicular to its axis, the ray tracing is the same as that observed in the plane passing through the center of a sphere.

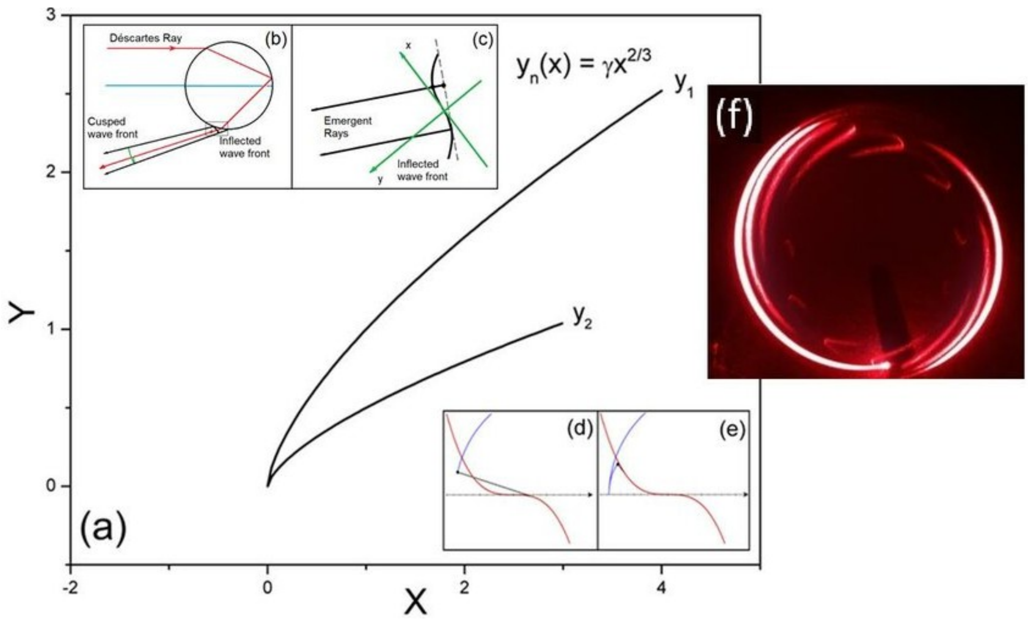

Figure 7 - Caustic formation (a) and its relationship with Cartesian radii (b), involutes (c)-(d)-(e) and the caustics in the luminous pattern obtained with the cylinder/laser system.

In relation to a transparent sphere, we notice the formation of a rainbow, which resembles the creation of caustics in the cylinder/laser system. When a raindrop acts as a lens, there are folds of light inside it that overlap to form a caustic. We can demonstrate the existence of these two overlapping branches in the case of a spherical drop projecting a beam of light onto a screen deflected by a glass bowl filled with water, particularly when the beam hits the circumference of the bowl near the point where the caustic occurs.

When a laser beam is incident obliquely on the glass cylinder, the Spiralbow displays similar patterns, but breaks the degeneracy of the folds in the light wave, resulting in the separation of the overlapping branches, just like in the case of a raindrop. These patterns reveal two arcs with a common point. This phenomenon illustrates how the cylinder/laser system helps us to understand the present in the rainbow. As far as we know, this observation is unique and has not been demonstrated in this way before.

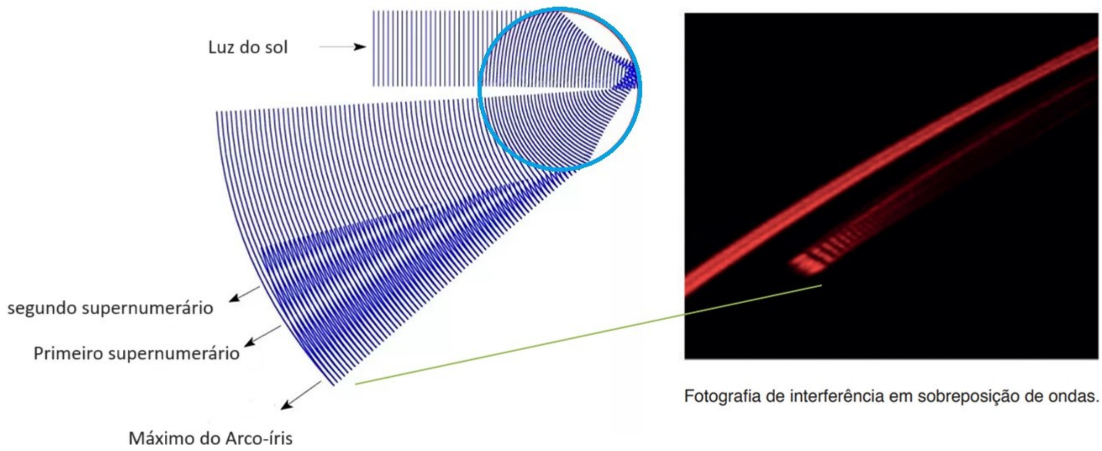

Figure 8 - Caustic interference effect in the laser/cylinder system.

For more details on the statements discussed in this topic we suggest Reference 9.

5. Topology

The links with mathematics don't stop there. We have mappings in this system that are related to Möbius transformations. Because we have non-linear transformations, this can involve non-Euclidean geometries that can be studied using complex analysis. In this case we need to understand the surfaces involved topologically as "topological varieties", which can have three types of curvature: negative, zero or positive, as we find in prismatic optical systems with spheres, cylinders, parallelepipeds, hyperbolic prisms and Beltrami pseudospheres.

In this way, we can apply Möbius transformations as complex mappings. Möbius transformations have applications in optics, particularly in the modeling and analysis of optical systems such as lenses and spherical mirrors. These transformations are used to describe how light rays propagate and interact in complex optical systems. Some ways in which Möbius transformations are related to optics are lens and mirror transformations: Möbius transformations can be used to model the properties of lenses and mirrors in optics, as they describe how light rays are refracted or reflected when passing through these systems. This is particularly useful when analyzing complex optical systems in which the geometry of the lenses and mirrors is complicated. Projection transformations: In optics, projection transformations are often used to represent how objects are projected onto a surface, such as a screen. The

Möbius can be used to describe these projections, especially when dealing with non-linear or curved optical systems.

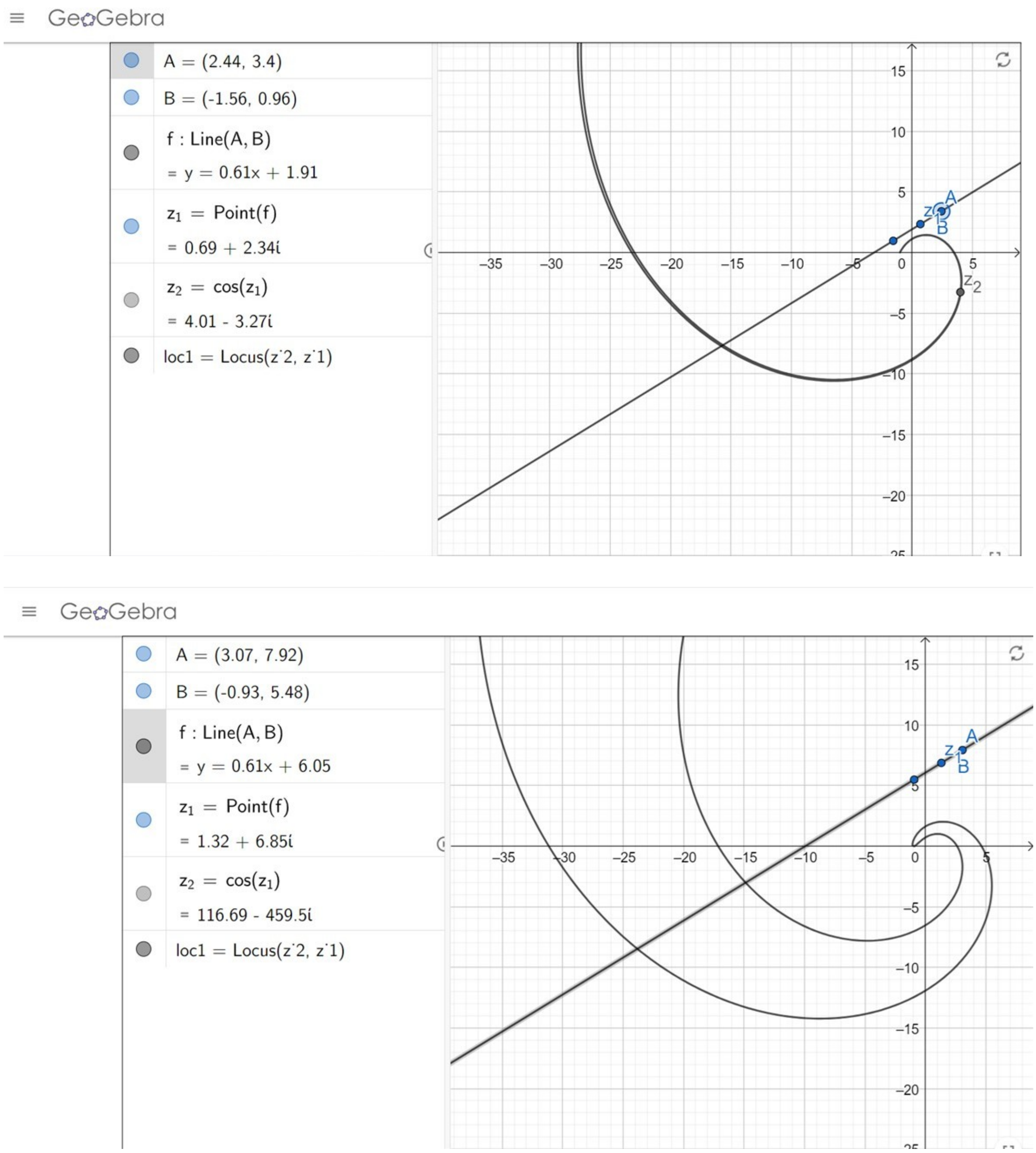

Figure 9 - Complex transformation cos(z) of a line into spirals.

https://www.geogebra.org/m/XEzcUc3H

Using Geogebra, we have the image of the graph of a line with a complex transformation cos (z). In the first image, the straight line in a complex plane is transformed into a spiral with the cos (z) function. By changing the location of the line in the second case, the spiral opens up into two branches, analogous to the case where we have the formation of involute curves.

Another application of these transformations is in the analysis of optical aberrations, which are unwanted distortions in the formation of images in optical systems. They help to understand how aberrations affect the propagation of light and the formation of images. In addition, another application of these transformations is in non-linear optical systems, where Möbius transformations can be used to describe how the intensity and phase of light change when interacting with non-linear media, such as non-linear crystals. This is essential for understanding non-linear optical phenomena such as frequency generation, wave mixing and other non-linear optical processes. In general, Möbius transformations are a powerful mathematical tool that can be applied to the analysis and modeling of complex optical systems. They play an important role in optics, helping to understand how light behaves and interacts in different optical contexts, including lenses, mirrors, projections and non-linear phenomena, such as what occurs in the cylinder/laser system, because when part of the laser beam is scattered by the cylinder and projected onto a screen, it results in the transformation of a simple straight line segment into an intriguing spiral pattern.

6. Dynamical Systems and Chaos

We can approach light in two different ways, either as a wave or as a ray, and the choice between these approaches depends on the scale and characteristics of the phenomena being studied. In many cases, both approaches can be used in a complementary way to obtain a comprehensive understanding of the behavior of light in various contexts. For the case of geometric optics, which adopts the ray model, treating light as discrete particles or rays of light propagating in a straight line, we can make an analogy with the mechanical case of particles scattered by a gravitational potential in space. This model proves useful for describing the behavior of light in situations where the wavelengths of light are significantly smaller than the dimensions of the objects involved, as in optical experiments with lenses, mirrors, prisms, among others. With this approach, some ray paths in optics can be related to hyperbolic dynamical systems. A dynamic system is considered hyperbolic when, in general terms, the tangent space above the asymptotic part of the phase space is divided into two complementary directions. One of these directions is characterized by contraction, while the other is characterized by expansion as the system evolves. This is similar to processes in which stretching and bending occur, or in other words, a hyperbolic dynamic system is one in which the trajectories in phase space can converge or diverge, and the two complementary directions exhibit opposite behaviors. This property is essential for understanding the stability and long-term behavior of dynamic systems, and plays a fundamental role in Chaos theory and non-linear dynamics. Thus, considering the cylinder as a scattering center for light rays in three-dimensional space, we can apply techniques from Chaos theory to the trajectory of particles represented by the light beam, which can present sensitivity to initial conditions due to the stretching and folding process, the so-called butterfly effect. We also have the formation of light patterns related to Arnold tongues using the representation of the circle map, as is done in circular billiards. We have already studied this butterfly effect in another optical system, which was the case of the hyperbolic prism, with the formation of self-similar patterns such as those that occur in Poincaré disks using Möbius transformations in the reflection in spherical mirrors [Ref. Hyperbolic Prism]. In this way, we understand that the interaction of light with a glass cylinder can lead to complex and chaotic phenomena, in which small variations in the initial conditions or in the geometry of the system can result in intricate and unpredictable light patterns. This relates to chaos theory and the concepts of sensitivity to initial conditions, or to other terms specific to the area of dynamic systems, such as the scenario known in English as "devil's staircase", which refer to the mathematical techniques and structures associated with describing the deviations of the laser beam by the glass cylinder.

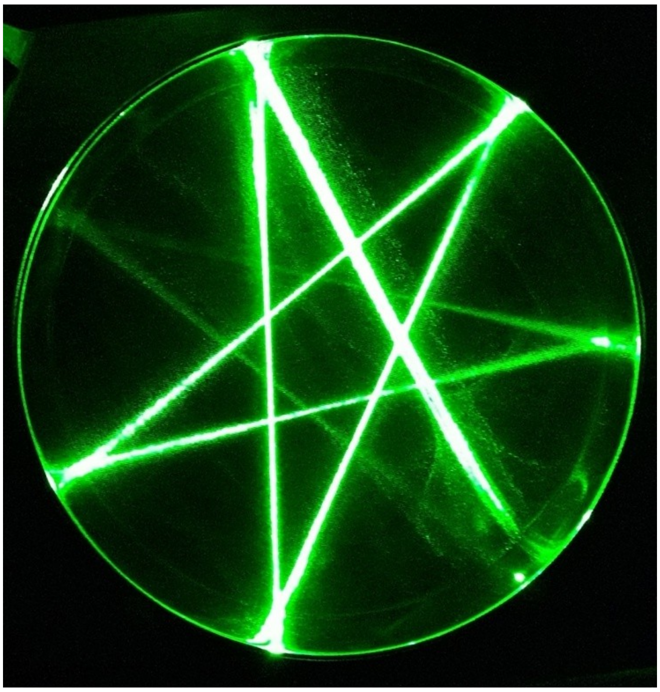

Figure 10 - Multiple reflections of a laser beam inside a cylinder.

In addition to the small deviations of the trajectory implying sensitivity to the initial conditions of the smooth hyperbolic dynamic systems described above, we have the case of the abrupt variation of the particle due to the refraction of light, where one part of the particles refracts and another part of the particles reflects. We thus have two cases in this cylinder/laser system: the first linked to sensitivity to initial conditions which means that small variations in the initial conditions of the "light particles" can result in significant changes in the behavior of the system in the long term. For example, if initially the particles in an analogous mechanical system have slightly different directions, these small differences will magnify as they propagate through the optical system, leading to different results. The second case is related to the abrupt interaction of light at the interface. When light passes from a medium to another, such as from air to glass, part of the light particles is refracted, which means that their direction changes as they enter the new medium. However, another part of the particles is reflected, with a change of direction opposite to that of the new medium. direction of the incident beam. This is common at interfaces between with different refractive index media, such as lenses or prisms.

These two phenomena, sensitivity to initial conditions and abrupt changes in ray trajectories, are fundamental aspects to be considered when analyzing the trajectory of light particles in complex optical systems. They can result in unpredictable and complex behavior as the light particles propagate through the optical system.

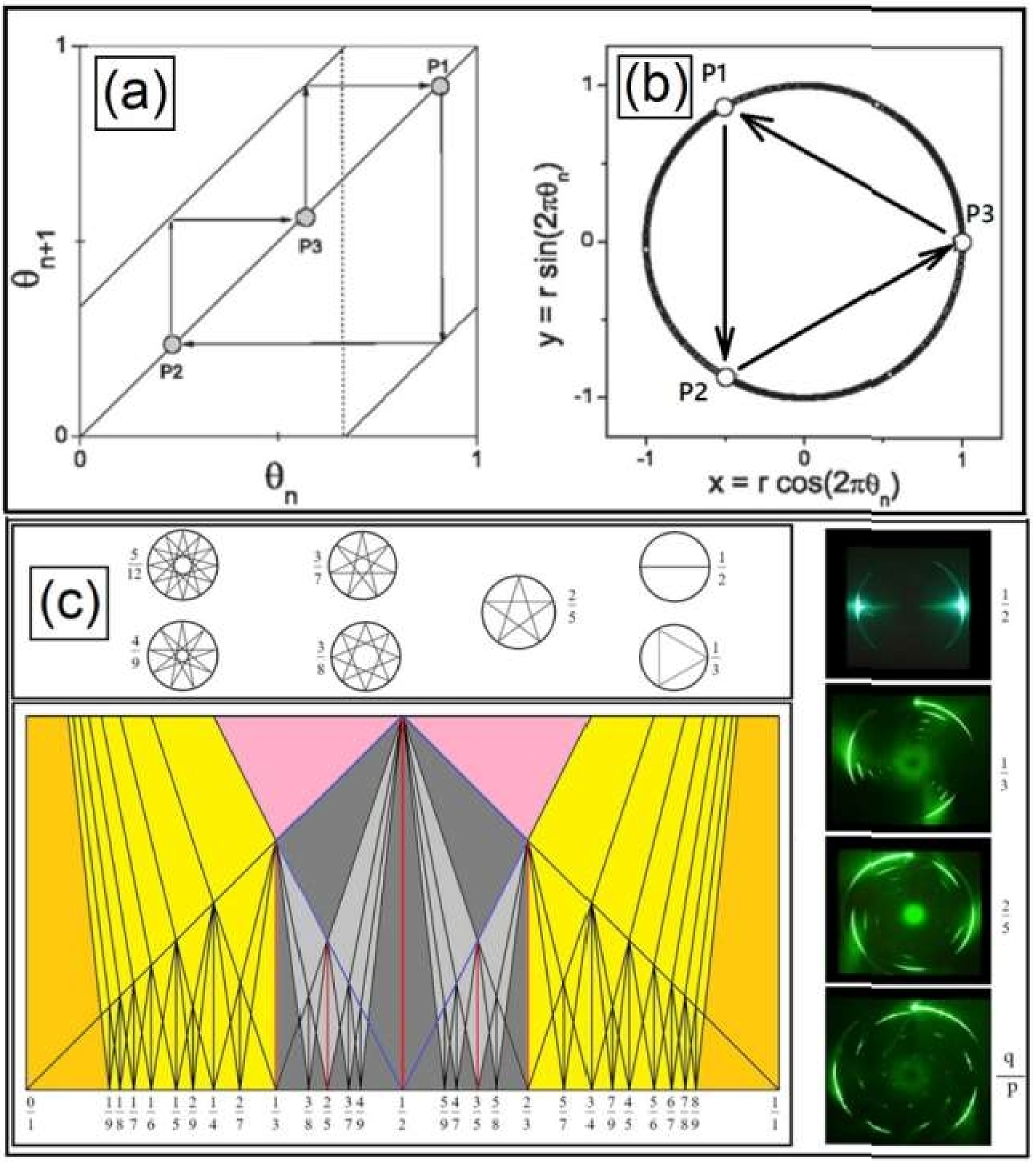

Figure 11 - Map of the circle (a), dynamics of a particle on a circle, (c) trajectories of a particle on a circumference, Arnold's languages with tree of Farey and patterns obtained with the cylinder/laser beam and its Farey mediants.

7. General Relativity

Our research also covers a case involving general relativity. We explore light-scattering centers, called gravitational lenses within the context of relativity. These gravitational lenses play a crucial role in the study of images of distant galaxies that undergo modifications due to relativistic effects. A striking example of these effects can be seen in Einstein's rings, which can be created by using lenses in the shape of a pseudosphere.

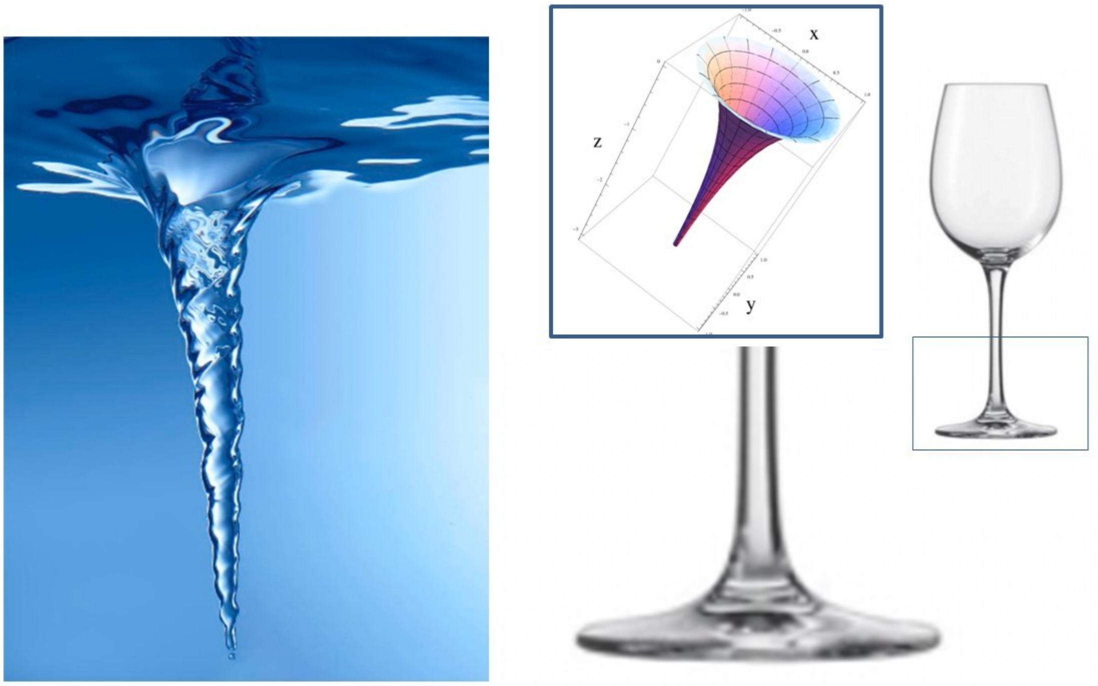

Figure 12 - Physical examples of optical systems analogous to the Beltrami pseudosphere.

The pseudosphere is a two-dimensional surface that can be represented in various ways, including by means of projections in three-dimensional space. It can be imagined as similar to a "bowl foot" or the smooth funnel-shaped surface of water that forms when water flows down a drain. It plays an important role in understanding non-Euclidean geometries and has applications in fields ranging from pure mathematics to theoretical physics. Beltrami's pseudosphere, also known simply as a pseudosphere, is a geometric shape with constant curvature, but unlike an ordinary sphere, its curvature is negative. This characteristic makes it a remarkable example of a surface with negative constant curvature. The properties of the pseudosphere are fascinating and are often used to illustrate concepts related to non-Euclidean geometry, where the sum of the angles of a triangle can be less than 180 degrees. The pseudosphere, due to its deformation, serves as an analogy for a massive body, such as a black hole, deflecting light from remote galaxies. This deviation results in the formation of light ring patterns around these massive objects along their trajectory.

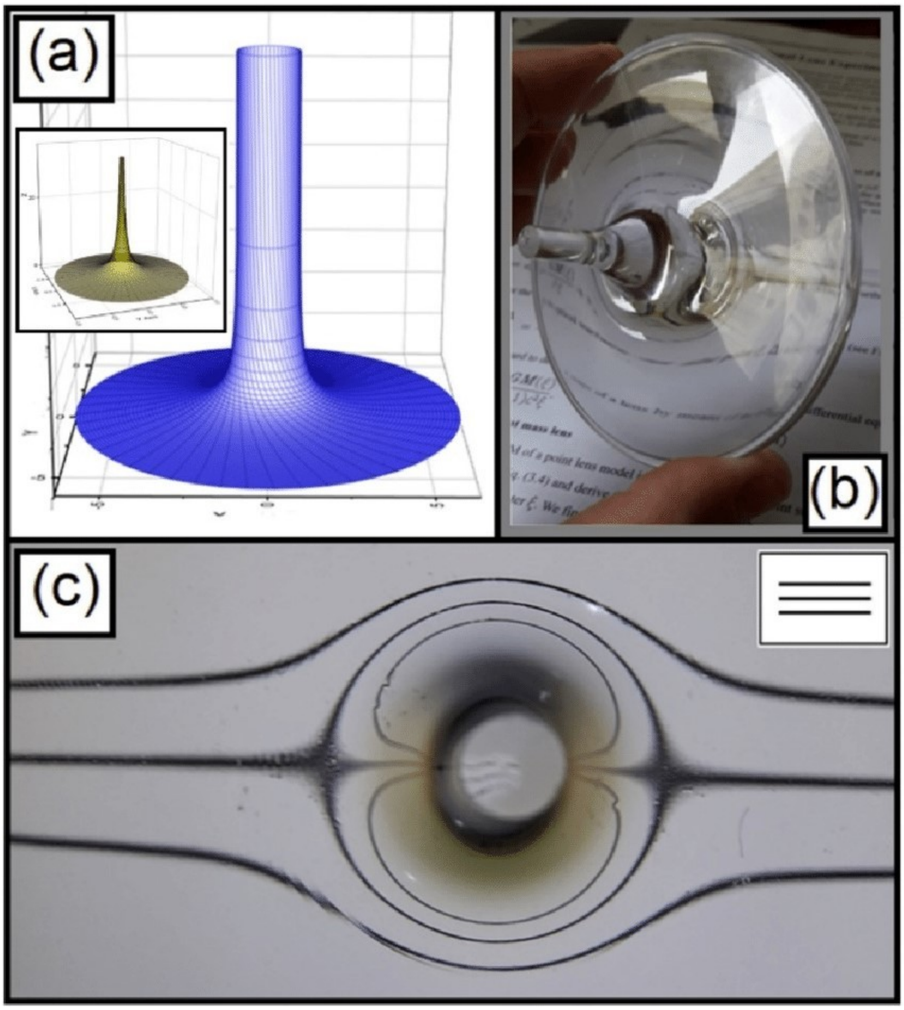

Figure 13 - The lens made from the foot of a wine glass: (a) cylinder deforming into a Beltrami sphere, (b) the foot of a wine glass lens, (c) three parallel lines seen through this lens.

The concept that connects geometric optics and relativity is Fermat's principle. This principle states that light follows the path of least time when passing from one point to another. In other words, when light propagates from point A to point B, it chooses a path that minimizes the time needed to travel that distance. As far as general relativity is concerned, Fermat's Principle can be interpreted in a more abstract way. Here, it's not just about paths of least time in the classical sense of geometric optics, but about geodesic paths in curved space-time. Instead of focusing on the speed of light in a single medium, general relativity deals with the curvature of space-time caused by the presence of mass and energy. In this way, light moves in geodesics, which are the shortest paths, i.e. the paths of least "length" in space-time, possible in a curved space-time. Fermat's Principle in general relativity implies that light follows geodesics in space-time, which are determined by the metric of this space-time.

time and the initial conditions of the movement related to the light trajectory. Thus, while in geometric optics Fermat's Principle relates to paths of least time in Euclidean space, in general relativity it relates to geodesics in curved space-time, representing the way light moves under the influence of gravity. Both interpretations are based on the idea of minimizing some kind of "time," but the specific application and interpretation varies between these two contexts, which is why we speak of an analogy between lenses formed with the laser cylinder system and gravitational lenses in general relativity.

Figure 14 - Rectangular tile pattern seen through a water vortex.

Other lenses based on Beltrami pseudo-spheres can be used to exemplify the optical analogy and general relativity, such as the image of water flowing down a pool drain. This image illustrates the distortion of a rectangular pattern at the bottom of a swimming pool caused by the conical shape of water collecting above a drain. The distorted image serves as an analogy for understanding the concept of general relativity in relation to optics.

In optics, distortion is due to the phenomenon of refraction, where light changes direction when passing from one medium to another with different densities, such as from air to water. This results in a curve in the path of the light rays, which in turn causes distortion of the image.

In general relativity, the analogy is made with the distortion of space-time caused by the presence of a massive object, such as a star, a galaxy or a black hole. Just as water bends the path of light in a swimming pool, the presence of mass bends space-time around it. This curvature of space-time is responsible for deviations in the trajectory of light, known as gravitational lenses. In both cases, optics and general relativity, light suffers deviations due to phenomena associated with changes in the geometry of space, either through the refraction of light in optics or the curvature of space-time in general relativity.

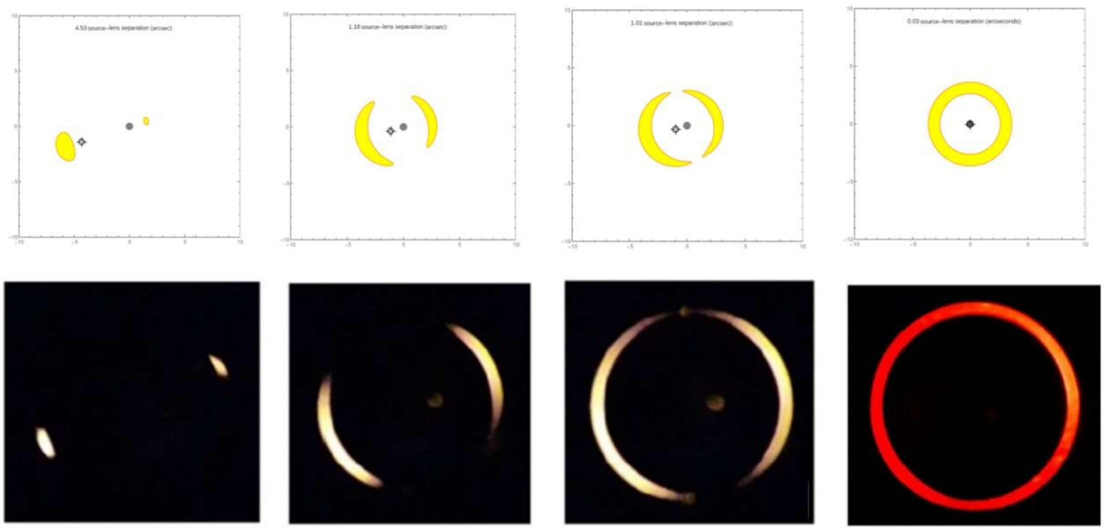

Figure 15 - Simulation of gravitational lenses and images of a luminous point obtained with the bowl-foot lens.

https://demonstrations.wolfram.com/GravitationalLensingByAPointMass/

In the figure we see a simulation and an experiment Light rays
from a source passing through a mass are bent due to the gravity of the mass, causing a change in the shape and size of the observed image. This effect is called gravitational lensing and, among other applications, is used to infer the
lens mass. This Demonstration shows the image created by a mass lens
point source. The original source is a point source of light at a fixed distance of light years from the observer.

To make the study of gravitational lenses easier to understand, we can use wine glasses as a visual analogy to understand and explore the effects of gravitational lenses.
gravitational lenses, associated with gravity, with optical lenses, such as lenses in cameras or glasses, as we did with a cup foot simulating a pseudo lens.
Beltrami's spherical lens, or even considering the contents of the glass as a lens. By simulating a wine glass to investigate visual parameters, we can make different combinations of these parameters, simulating various types of lens effects. For example, with the demonstration shown below, in which the left panel shows the wine glass from a non-orthographic perspective, and the right panel shows a flat projection of the base of the wine glass, it provides a way of simulating the contents of the wine glass.

visual to explore and understand these lens effects using the analogy of the wine glass.

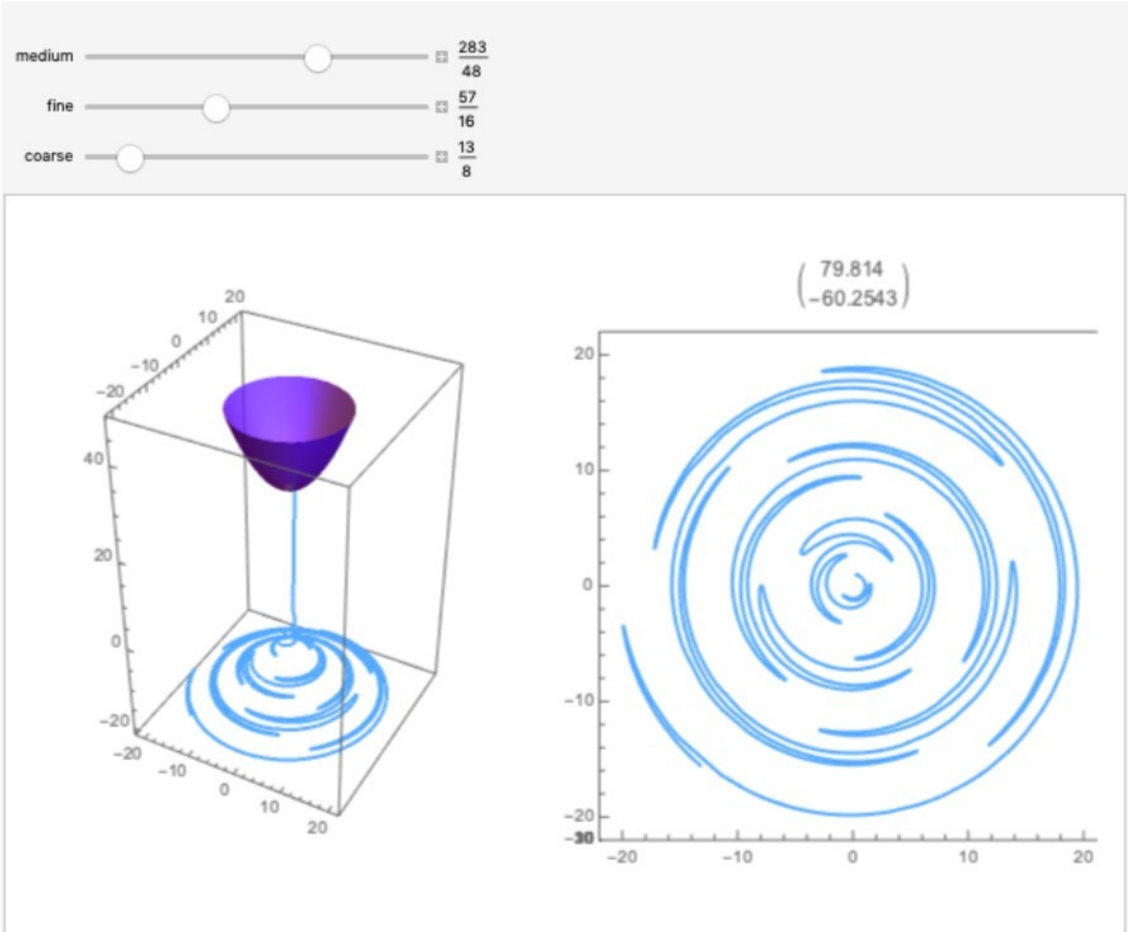

Figure 16 - Simulation for a gravitational lens.

https://demonstrations.wolfram.com/WineGlassLensing/

In this image with spiral lines, we can see the formation of curves similar to the involute curves discussed in the section on rainbows and catastrophes.

8. Conclusion

In this text, we summarize how we explored some mathematical aspects of light propagation in optical systems with a laser beam incident on cylindrical lenses. By relating this system to the formation of rainbows, we try to show how this optical system is connected to phenomena in our daily lives. The interdisciplinarity of these topics discussed above, combining diverse fields such as topology, optics, dynamical systems and relativity, makes it scientifically fascinating. This interconnectedness presents complex challenges, but also opens doors to innovative research opportunities. What's more, the complexity intrinsic to the interaction between these fields can lead to significant discoveries and practical applications in areas such as technology and astronomy. By exploring complex phenomena related to these topics, we broaden our fundamental knowledge of the universe and expand the frontiers of scientific research.

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